Enlarged CG and two-level preconditioner for the Cosmic Microwave Background

Thibault Cimic, Laura Grigori, Jan Papež, and Olivier Tissot

thibault.cimic@inria.fr INRIA Paris, team ALPINES

Introduction

Studies of the Cosmic Microwave Background (CMB) anisotropies have been driving the progress in our understanding of the Universe for nearly 25 years. The current and forthcoming CMB observatories are expected to deliver unprecedented insights about the Universe's beginning and evolution, producing enormous data sets, ($O(10^{15})$) and thus calling for advanced, high performance data analysis techniques.

Map-making problem

Settings:

- number of pixels $n_p \sim O(10^6)$
- number of measurements $n_t \sim O(10^{15})$
- $P \in \mathbb{R}^{n_t \times 3n_p}$, $N^{-1} \in \mathbb{R}^{n_t \times n_t}$, $m_{ML} \in \mathbb{R}^{3n_p}$, $d \in \mathbb{R}^{n_t}$

$$\underbrace{d}_{\text{observed data}} = \underbrace{P}_{\text{pointing matrix}} \cdot \underbrace{m}_{\text{unknown map}} + \underbrace{n}_{\text{noise}}$$

$$\Rightarrow (P^T N^{-1} P) m_{ML} = P^T N^{-1} d \qquad (1)$$

A few comments:

- noise is very large, its inverse covariance matrix is known a priori and it is of special form
- the system matrix cannot be formed but we can perform matrix-vector product with it
- the application of matrix-vector product involves FFTs and it is very time-costly

Techniques

The state-of-the-art solver is **Preconditioned Conjugate Gradient** (PCG) ($O(100)$ iterations) with:

- one-level preconditioner [4]
  block-diagonal prec.: $P^t \text{diag}(N^{-1})P$
- two-level preconditioner [2]
  block-diagonal prec. + deflation of "smooth" eigenvectors

The techniques under study:

- **Enlarged Conjugate Gradient** (ECG)
- Two-level **GenEO** preconditioner from Domain Decomposition methods

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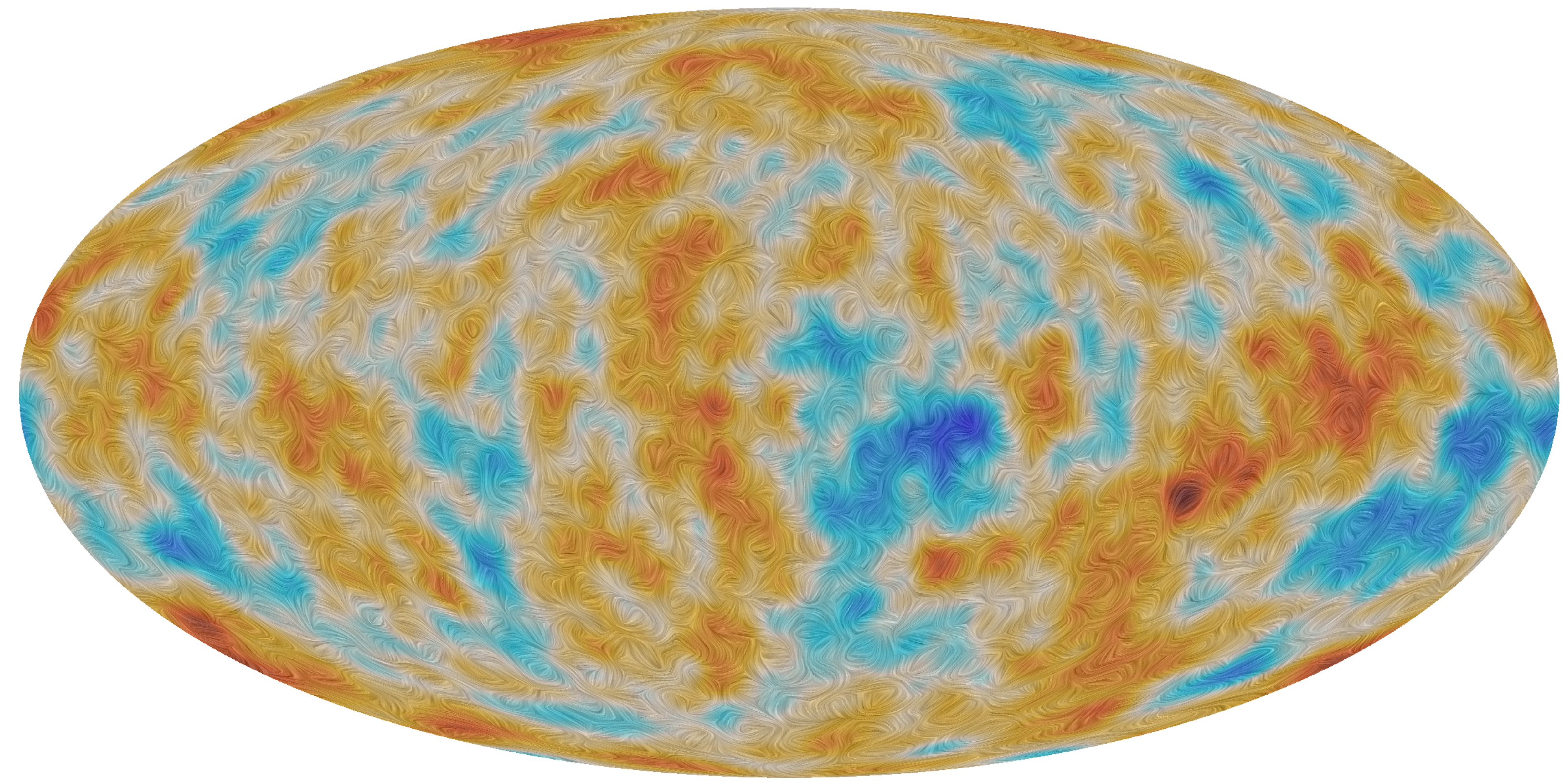

Figure 1:A visualization of the polarization of CMB as detected by ESA's Planck satellite over the entire sky, www.jpl.nasa.gov.

Enlarged conjugate gradient

Enlarged Krylov techniques are projection methods on a Krylov subspace enlarged by a parameter $t \in \mathbb{N}$ to solve linear systems: $A\underline{x} = b$. By initially splitting the residual r_0 into t linearly independant vectors, a set of t new vectors is added to the enlarged Krylov sudbspace, $K_{k,t}(A, r_0)$, at each iteration k. The k-th ECG approximation then satisfies:

$$||x_k - \underline{x}||_A = \min_{x \in x_0 + K_{k-1,t}} ||x - \underline{x}||_A \qquad (2)$$

$$K_{k-1} \subset K_{k-1,t} \qquad (3)$$

As a result, a new convergence rate can be established:

$$||x_k - \underline{x}||_A \leq 2||\hat{e}_0||_A \left(\frac{\sqrt{\kappa_t} - 1}{\sqrt{\kappa_t} + 1}\right)^k \qquad (4)$$

with $\kappa_t = \lambda_n / \lambda_t$ and $\hat{e}_0 := E_0(\phi_1^t E_0)^{-1}$, where ϕ_1 denotes the t eigenvectors associated to the t smallest eigenvalues: $\lambda_1 \leq ... \leq \lambda_t$, and E_0 is the initial enlarged error.
This results in a trade-off between the decrease of the number of iteration needed to reach a certain threshold and the increase of the cost of each iteration, illustrated in [3]:

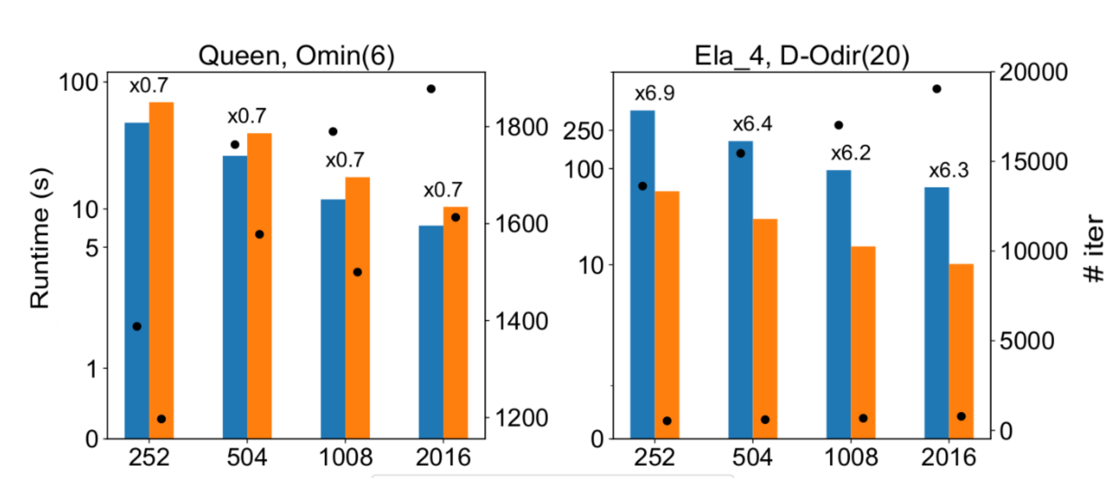

Figure 2:Queen: structural problem matrix, Ela_4: Linear Elasticity matrix - left y-axis is the total runtimes (columns), right y-axis is the iteration number (dots), x-axis is the number of MPI processes

GenEO preconditioner

In domain decomposition, a two-level preconditionner is build from the classical Additive Schwartz (AS) one-level preconditionner by adding a coarse space correction. This typically makes use of an underlying setting from a PDE, such as with Generalized Eigenvalue in the Overlap (GenEO) preconditioner.
In [1], the authors describe a fully algebraic two-level domain decomposition preconditionner, by defining an algebraic local SPSD splitting (ALS), $\tilde{A}_i$, of a matrix A w.r.t. a subdomain $i = 1, ..., N$:

$$\mathcal{P}_i \tilde{A}_i \mathcal{P}_i^t = \begin{pmatrix} \mathcal{R}_{i,0} A \mathcal{R}_{i,0}^t & \mathcal{R}_{i,0} A \mathcal{R}_{i,\delta}^t & \\ \mathcal{R}_{i,\delta} A \mathcal{R}_{i,0}^t & \tilde{A}_\delta & \\ & & 0 \end{pmatrix} \qquad (5)$$

s.t. $\forall u \in \mathbb{R}^n$,

$$0 \leq u^t \tilde{A}_i u \leq u^t A u \qquad (6)$$

Then a coarse space is build by solving generalized eigenvalues problem on each subdomains i with $\tilde{A}_i$. Numerical results are shown in Table 1.
In our problem (1), not originating from a PDE setting, we would like to build a coarse space accounting for the smallest eigenvalues of the operator. For that, we can make use of a similar splitting:

$$P^t N^{-1} P = \sum_{i=1}^{n_{blc}} P_i^t N_i^{-1} P_i \qquad (7)$$

which is an SPSD splitting of the matrix but not necessarily local, neither in the time domain, nor in the pixel domain. Therefore the generalized eigenvalue problems considered in [1] cannot be derived from this splitting, and new ones must be considered.

Numerical results

We first show some results on the ECG applied to the map-making problem:

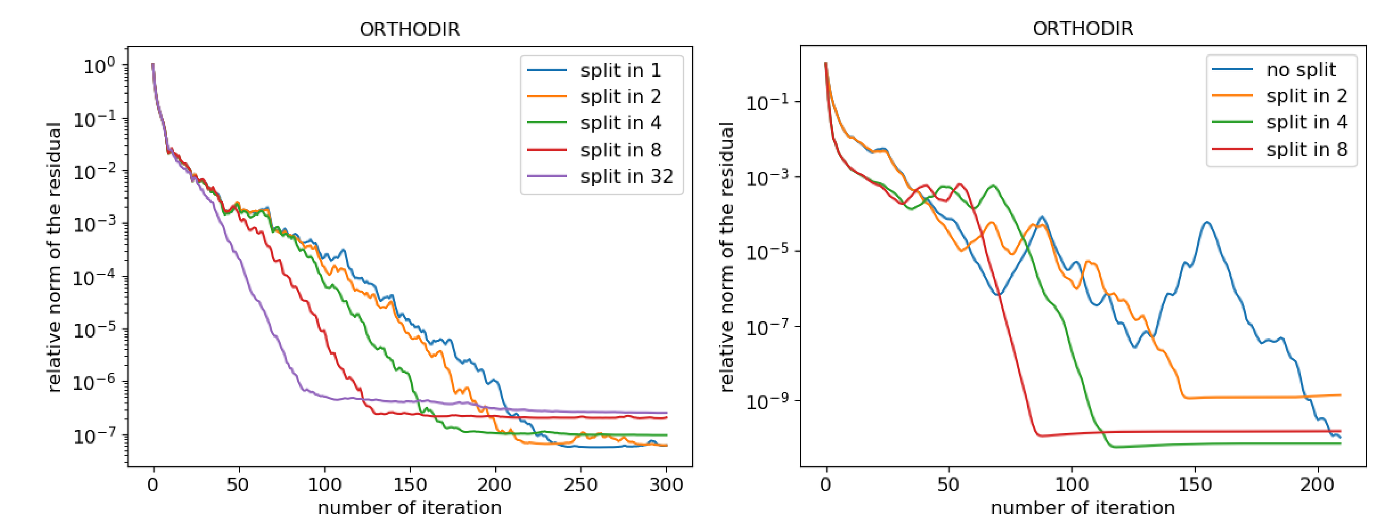

Figure 3:Two different map-making test cases solved with ECG and block-diagonal preconditioner

And numerical results from [1] with an 3D elasticity problem:

N	dim_{uC}	n_{uC}	dim_{α_1}	n_{α_1}	dim_{α_2}	n_{α_2}	dim_{Gen}	n_{Gen}
4	82	20	92	19	120	18	106	20
8	179	23	209	20	240	20	229	24
16	304	37	394	30	480	28	391	38
32	447	53	583	45	960	36	614	42
64	622	84	769	73	1920	51	850	55
128	969	131	1096	114	3834	77	1326	61

Table 1:Dimension of the coarse (dim), and the iteration counts (n) for various two-level preconditioners and varying number of subdomains N. Best ALS preconditioner (uC), GenEO precondtioner (Gen), and convex combinations of ALS preconditioners (α_1, α_2).

On-going work

- Fully understanding of the trade-off between the number of iteration and the splitting parameter t in enlarged projection methods.
- Building a fully algebraic domain-decomposition-like two-level preconditionner for the map-making problem.

References

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